**Back-Paper Exam**
**Algebra-IV**
**B. Math - Second year**
**2014-2015**

Time: 3 hrs
Max score: 100

Answer all questions.

(1) Construct a field $F$ with 9 elements and compute $Gal(F|\mathbb{Z}_3)$. (5)

(2) (i) Let $f(x) \in F[x]$ be a polynomial of degree $n$. Let $K$ be its splitting field. Show that $[K : F]$ divides $n!$.
(ii) Determine the splitting field of $x^4 + 2$ over $\mathbb{Q}$. (8+5)

(3) (i) State the fundamental theorem of Galois theory.
(ii) Compute the Galois group of the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. (6+8)

(4) Show that if $L_1$ and $L_2$ are Galois over $K$ in a common field then $[L_1L_2 : K] = [L_1 : K][L_2 : K]$ if and only if $L_1 \cap L_2 = K$. More generally, show that there is an embedding $Gal(L_1L_2|K) \to Gal(L_1|K) \times Gal(L_2|K)$ and it is an isomorphism if and only if $L_1 \cap L_2 = K$. (8+8)

(5) Let $f(x) \in F[x]$ be a separable polynomial of degree $n$. Show that $f(x)$ is irreducible in $F[x]$ if and only if its Galois group over $F$ is a transitive subgroup of $S_n$. (10)

(6) (a) Define discriminant $D_f$ of a polynomial $f(x)$.
(b) Let $char(F) \neq 2$, and let $f(x) \in F[x]$ be a separable polynomial of degree $n$. Show that the Galois group of $f(x)$ over $F$ is a subgroup of the alternating group $A_n$ if and only if $D_f$ is a square in $F$. (4+10)

(7) Define a cyclic extension. Show that any cyclic extension of degree $n$ over a field $F$ of characteristic not dividing $n$ which contains the $n^{th}$ roots of unity is of the form $F(\sqrt[n]{a})$ for some $a \in F$. (3+10)

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(8) (i) Define root extension of a field $F$.
(ii) Let $K|F$ be a root extension. Then show that there exists an extension $L|K$ such that $L|F$ is a Galois root extension. (4+ 12)

(9) Define finite solvable groups.
Show that the polynomial $f(x)$ can be solved by radicals if and only if its Galois group is a solvable group. (3+12)